Math 174E, Mathematics of Finance

Fall 2016

Final Exam

Name:

UID:

There are ten problems with a total of 100 points.

Problem 1: Assume that a treasury bond with 1.5 years until maturity has a par value of 100, a coupon of 2% with semi-annual payments, and a current price of 97.

(a) Set up an equation for the yield y of the bond (with continuous compounding) in the form $f(y) = 0$. (3pts)

(b) If you want to compute y by using Newton's iteration method, which approximate value y_0 for y would you take as the initial value for the iteration? Hint: This value should take the difference of the par value and the current price into account. (3pts)

(c) Based on Newton's iteration method express the next approximate value y_1 for y in terms of y_0 (do not replace y_0 by the explicit numerical value found in (b)). (4pts)

Problem 2: We consider a call option on a stock. The current price of the underlying is $S_0 = 40$ and the strike of the option is $K = 38$. The time to expiration is two weeks. We want to use a two-step binomial model to get a rough estimate on the fair price of the call. We neglect interest rates and estimate that the stock price goes up or down by 5% each week.

(a) Draw the graph that represents the two-step binomial model in this case, where at each node of the graph you indicate the stock price and the fair value of the call. (6pts)

(b) What is the current fair value c of the call? (1pt)

(c) Suppose we want to compare the value c in (b) with the value obtained from the Black-Scholes formula. Which values for σ and T should we take here? (3pts)

Problem 3: We consider several call and put options on TWTR with expiration Dec. 16, 2016. The last price of the stock on Dec. 02, 2016, was $S_0 = 17.93$, and we had the following last traded prices for the options:

call	strike	put
1.21	17	0.26
0.95	17.5	0.45
0.64	18	0.68
0.45	18.5	1.00

We can neglect interest rates and note that TWTR does not pay dividends.

(a) There are two arbitrage opportunities where you can make a gain of at least 0.05 (per share) with a suitable combination of positions. Describe the strategy and the gain in each case! (6pts)

(b) One option seems clearly overpriced. Which one is it and what should be an upper bound for its price? (2pts)

(c) Suppose TWTR paid a dividend of 0.10 per share before expiration of the options. Based on the same prices, how could you use the stock and the options with strike 17 for an arbitrage opportunity? With a suitable position, what would be your profit (per share)? (2pts)

Problem 4: Suppose the spot price for an investment asset is $S_0 = 40$. We consider a forward contract on the asset with delivery in one year. We assume that the risk-free rate for one year is 5% (with YEARLY compounding).

(a) What is the forward price F_0 of the asset? (3pts)

(b) Show that if the forward contract is priced higher than the value in F_0 in (a), then an arbitrage opportunity arises. Describe the arbitrage strategy in detail! (4pts)

(c) Suppose that briefly before maturity of the forward contract, the asset generates an income of $I = 4$. How does this affect the forward price F_0? (3pts)

Problem 5: (a) Define the "zero-rate" for a given number n of years.   (2pts)

(b) What does buying "stocks on margin" mean?   (2pts)

(c) Explain the concept of "dynamic Δ-hedging"!   (3pts)

(d) Suppose the (implied) volatility of a stock is 3% per week. What is its volatility σ per year?   (3pts)

Problem 6: According to the Black-Scholes model for option pricing the fair value c of a (European) call option is

$$c = S_0 N(d_+) - e^{-rT} K N(d_-),$$

where

$$d_\pm = \frac{\log(S_0/K) + rT \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and the parameters have the usual meaning. We know that the parameter Vega $\mathcal{V}$ of the option measures how a small change $\Delta\sigma$ of the implied volatility σ of the stock and a small change Δc of the option price are related; namely, $\Delta c = \mathcal{V}\Delta\sigma$.

(a) Express $\mathcal{V}$ explicitly as a derivative and show that $\mathcal{V} = S_0\sqrt{T}N'(d_+)$, where N denotes the cumulative distribution function of the standard normal random variable and N' its derivative. Hint: You can use the fact that $S_0 N'(d_+) = e^{-rt}K N'(d_-)$ without further justification. It can be verified by a straightforward, but somewhat lengthy and tedious computation. (4pts)

(b) Use (a) to justify the following fact: a larger implied volatility of the stock leads to a larger value of the call (with all other parameters the same). (3pts)

(c) Show that the same statement as in (b) is also true for put options. (3pts)

Problem 7: We consider call options O_1, O_2, O_3 with the same expiration on some underlying stock. We have the following strikes and prices for the options:

option	strike	price
O_1	140	22
O_2	150	17
O_3	160	14

(a) Which are the three different bull spreads that you can form with contracts on these options? (3pts)

(b) For each of the three bull spreads determine the stock price that maximizes the profit. (3pts)

(c) Suppose you invest $12,000 in one of the bull spreads. Create a table showing the number of contracts traded for each bull spread and the maximal profit. Which bull spread should you invest in to maximize your potential profit? (4pts)

Problem 8: You entered a covered call strategy holding a long position of 1000 shares of PG (=Procter & Gamble Inc.) and a short position of 10 contracts for the call on PG with strike 83.5 and expiration Jan. 06, 2017. We have $S_0 = 82.50$ for the stock price and $c = 1.10$ for the price of the call. Moreover, $\Delta = 0.4$ and $\Gamma = 0.1$ for the Greeks of the call. The numbers are rounded slightly for simplicity.

(a) Compute the total Δ of your portfolio and use this to estimate by how much the value of your portfolio will approximately change if the stock changes by ± 0.10.

(3pts)

(b) You speculate on the options expiring worthless, but you are concerned about a small downward trend of the stock price. One possibility is to sell a number of shares to make your portfolio insensitive to small changes of the stock price. How many shares would you sell if you followed this strategy? Justify your answer!

(2pts)

(c) As PG will pay a quarterly dividend of about 0.67 per share in late January, you actually prefer to keep your long position in PG and rather sell more call contracts. How many call contract could you sell in addition and keep a Δ-neutral position?

(3pts)

(d) Suppose you follow the strategy in (c), and the stock price goes up by 0.50. How many additional shares would you (approximately) have to buy to keep a hedged position? (2pts)

Problem 9: We consider the stochastic differential equation

$$\text{(1)} \qquad\qquad dX_t = \alpha X_t \, dt + \beta X_t \, dB_t$$

for a quantity X_t depending on time $t \geq 0$. Here α and β are positive constants, and $\{B_t\}_{t \in [0,\infty)}$ denotes Brownian motion. The purpose of this problem is to show that then X_t follows a geometric Brownian motion of the form

$$X_t = X_0 \exp((\alpha - \tfrac{1}{2}\beta^2)t + \beta B_t)$$

for $t \geq 0$. Complete the following steps:

(a) Consider a function $f(x,y) = \exp(Ax + By)$ at a point (x,y), where A and B are constants. By considering the Taylor expansion of f at (x,y) up to second order terms, find the (approximate) change df of the function f, if we move from (x,y) to a nearby point $(x+dx, y+dy)$. (4pts)

(b) Suppose that X_t can be expressed in the form

$$X_t = Cf(t, B_t),$$

where $f(x,y) = \exp(Ax + By)$ for some constants A, B, C. Express the change of dX_t in a small time interval dt in terms of dt and the change dB_t of Brownian motion in this time interval. Hint: Use (a) and the basic formulas of Itô calculus. (3pts)

(c) Find the constants A, B, C in (b) so that X_t solves the given SDE (1). Show your work! (3pts)

Problem 10: You are an investor and have two investment choices: a risk-free asset E and a riskier, but potentially more rewarding asset F. You choose the following strategy: you fix $p \in [0, 1]$ and at any moment you invest the proportion p of your portfolio in the risky investment F and the rest $1 - p$ in the risk-free investment E. We assume that the necessary reallocations of assets can be performed instantaneously without transaction costs. The purpose of this problem is to find the optimal proportion p that gives the largest growth of the value of your portfolio in the long run.

The first investment choice E gives the risk-free rate r as a return rate. So if E_t is the amount invested in E at time t, then the return dR_E on this investment in a small time interval dt is given by

$$dR_E = rE_t \, dt.$$

The second investment F gives a higher average return $q > r$, but has a non-zero volatility σ. So if F_t is the amount invested in F at time t, then the return dR_F on this investment in a small time interval dt can be modeled by

$$dR_F = qF_t \, dt + \sigma F_t \, dB_t,$$

where $\{B_t\}_{t \in [0,\infty)}$ is a Brownian motion.

(a) Let P_t be the value of your portfolio at time t. Express E_t and F_t and the corresponding returns dR_E and dR_F in terms of dt, dB_t, and P_t (and the given parameters). (1pt)

(b) By using (a) find the change dP_t of the value of your portfolio in a small time interval dt. Show that this leads to an SDE for P_t. (2pts)

(c) By using the results from Problem 9, show that

$$P_t = P_0 \exp\left(((1-p)r + pq - \tfrac{1}{2}p^2\sigma^2)t + p\sigma B_t\right).$$

(2pts)

(d) Which expression has to be maximized (over $p \in [0, 1]$) to ensure the largest growth of the portfolio as $t \to +\infty$ with "almost certainty". Justify your answer! Hint: We have $B_t/t \to 0$ almost surely as $t \to +\infty$. (1pt)

(e) Solve the maximization problem for p in (d). Hint: There are two cases! (2pts)

(f) Find the optimal value for p if $r = 3\%$, $q = 5\%$, and $\sigma = 20\%$. How does the result change if $\sigma = 10\%$? (2pts)